SPARSIFICATION OF SIMPLICIAL COMPLEXES
VIA NETWORK DENSITY OF STATES

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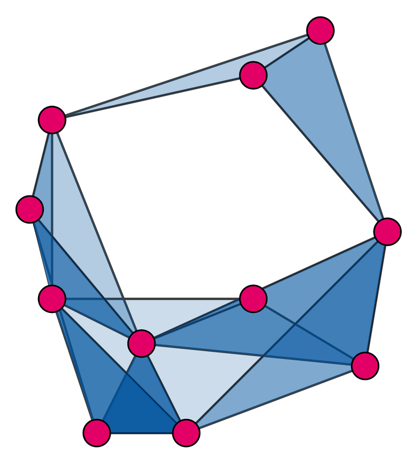
Simplicial complexes



Def. collection of simplices closed for face inclusion

$$\mathcal{K} = \Big\{ \underbrace{\mathcal{V}_0(\mathcal{K})}_{\text{nodes}},\ \underbrace{\mathcal{V}_1(\mathcal{K})}_{\text{edges}},\ \underbrace{\mathcal{V}_2(\mathcal{K})}_{\text{triangles}}, \ldots \Big\}$$

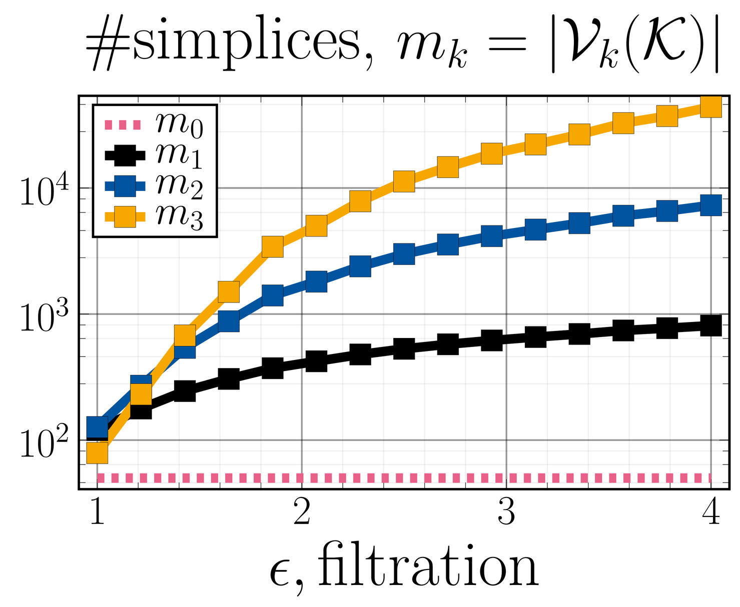
Boundary maps (simplex → boundary):

$$B_1 : \underbrace{\mathcal{V}_1(\mathcal{K})}_{\text{edges}} \to \underbrace{\mathcal{V}_0(\mathcal{K})}_{\text{nodes}}, \qquad B_2 : \underbrace{\mathcal{V}_2(\mathcal{K})}_{\text{triangles}} \to \underbrace{\mathcal{V}_1(\mathcal{K})}_{\text{edges}}$$

Hodge Laplacians:

$$L_k = \underbrace{B_k^\top B_k}_{\substack{\text{down-adjacent}\\ k\text{-simplices}}} + \underbrace{B_{k+1} B_{k+1}^\top}_{\substack{\text{up-adjacent}\\ k\text{-simplices}}} = L_k^{\downarrow} + L_k^{\uparrow}$$

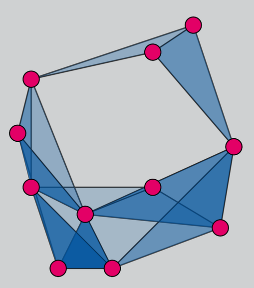
- shift operator for simplicial complexes
- encodes topological information
- used in topological data analysis, GNNs, random walks, etc.



Problem
- number of simplices m_k explodes
- expensive inference for GNNs
- expensive solution $L^\dagger$

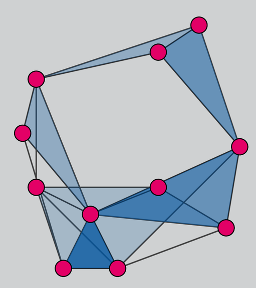
Sparsification

Original complex $\mathcal{K}$ is too dense!



same nodes, $m_0(\mathcal{L}) = m_0(\mathcal{K})$
same edges, $m_1(\mathcal{L}) = m_1(\mathcal{K})$

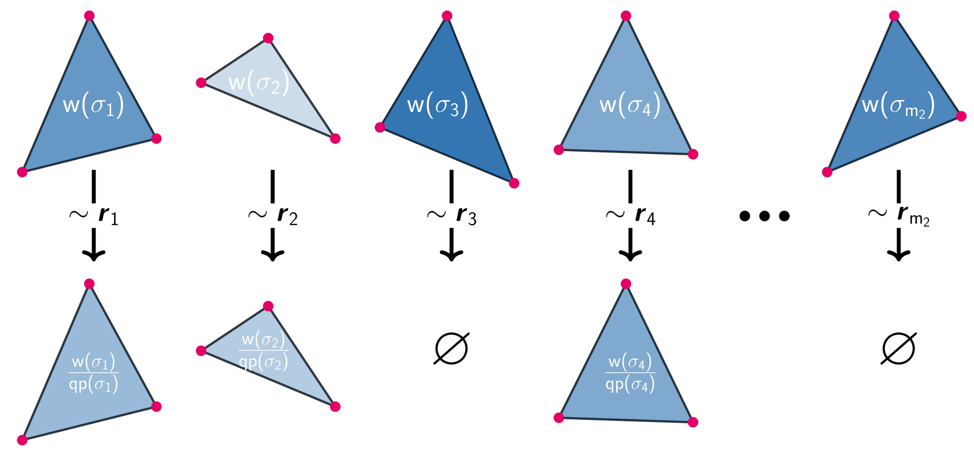
fewer triangles, $m_2(\mathcal{L}) \ll m_2(\mathcal{K})$
+
similar Laplacians
$\|L_1^{\uparrow}(\mathcal{L}) - L_1^{\uparrow}(\mathcal{K})\|_2 \le \varepsilon \|L_1^{\uparrow}(\mathcal{K})\|_2$



Spielman Spectral Sparsification

Sample $q(m_k) = \mathcal{O}(m_k \log m_k / \varepsilon)$ simplices according to GER

$$\boldsymbol{r} = \operatorname{diag}\left(B_k^\top \underbrace{(L_k^{\uparrow})^\dagger}_{\text{expensive}} B_k \right)$$



Problems with Direct Approach

- lack of efficient solvers for $\left(L_k^{\uparrow}\right)^\dagger$ for dense simplicial complexes ($m_{k+1} \gg m_k \ln m_k/\varepsilon$);
- poorly conditioned matrix $L_k^{\uparrow}$;
- computational cost $\mathcal{O}\left(m_k^3\right)$ and up to $\mathcal{O}\left(m_0^{3(k+1)}\right)$.

TL;DR Results

- you can sparsify any complex at any level with $\mathcal{O}\left(m_k^{1+\frac{4}{k+1}}\right)$ complexity;
- arbitrary controlled approximation error;
- limited to sparse **matrix multiplications** → small memory consumption.

arXiv code

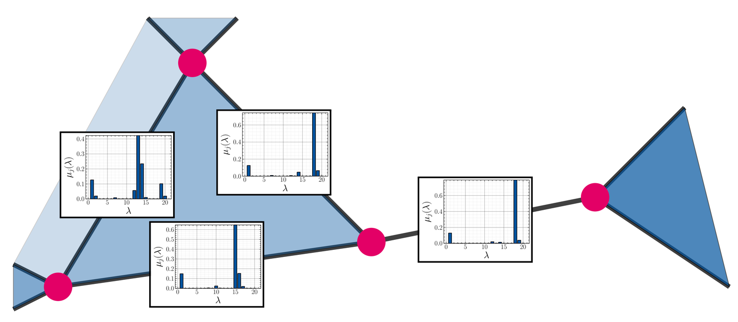
Novel approximation method

Let $\{\lambda_i, \boldsymbol{q}_i\}$ be eigenpairs of $L_k^{\uparrow}$.

- local density of state is

$$\mu_j(\lambda \mid L_k^{\uparrow}) = \sum_i |q_{ij}|^2 \delta(\lambda - \lambda_k)$$

- encodes spectral contribution of each edge/simplex;
- provides a functional description of the spectrum;
- still requires $\mathcal{O}\left(m_k^3\right)$ computation!



Thm

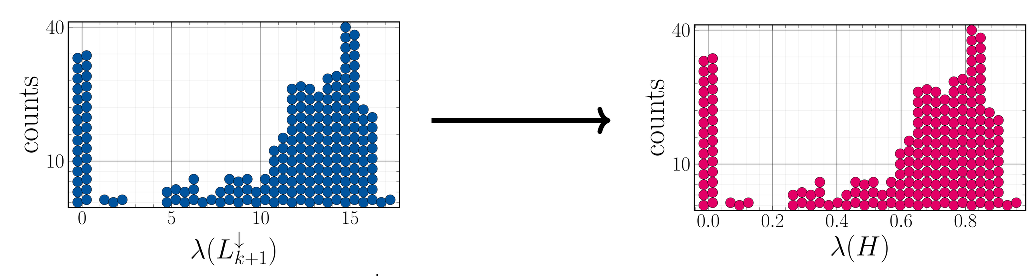
GER $\boldsymbol{r}$ for $L_k^{\uparrow}$ is defined in terms of the local spectral information:

$$\boldsymbol{r}_i = \int_{\mathbb{R}\setminus\{0\}} \mu_i(\lambda \mid L_{k+1}^{\downarrow}) d\lambda$$

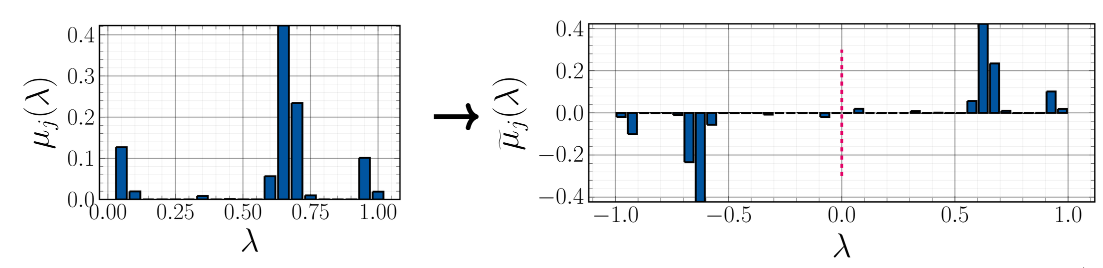
Can one avoid computing the eigenpairs?

Kernel Ignoring Decomposition

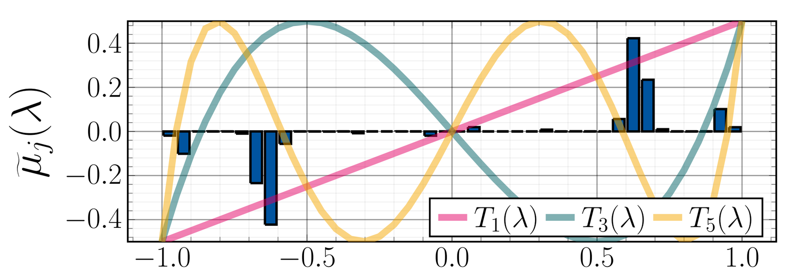
Idea: approximate LDoS $\mu_j(\lambda \mid L_k^{\uparrow})$ at every point except $\lambda = 0$.



1. scale $L_{k+1}^{\downarrow} \to H$ so $\sigma(H) \in [0, 1]$



2. exclude $\lambda = 0$ and symmetrize $\mu_j(\lambda \mid L_k^{\uparrow})$ to make an odd $\tilde{\mu}_j(\lambda \mid H)$



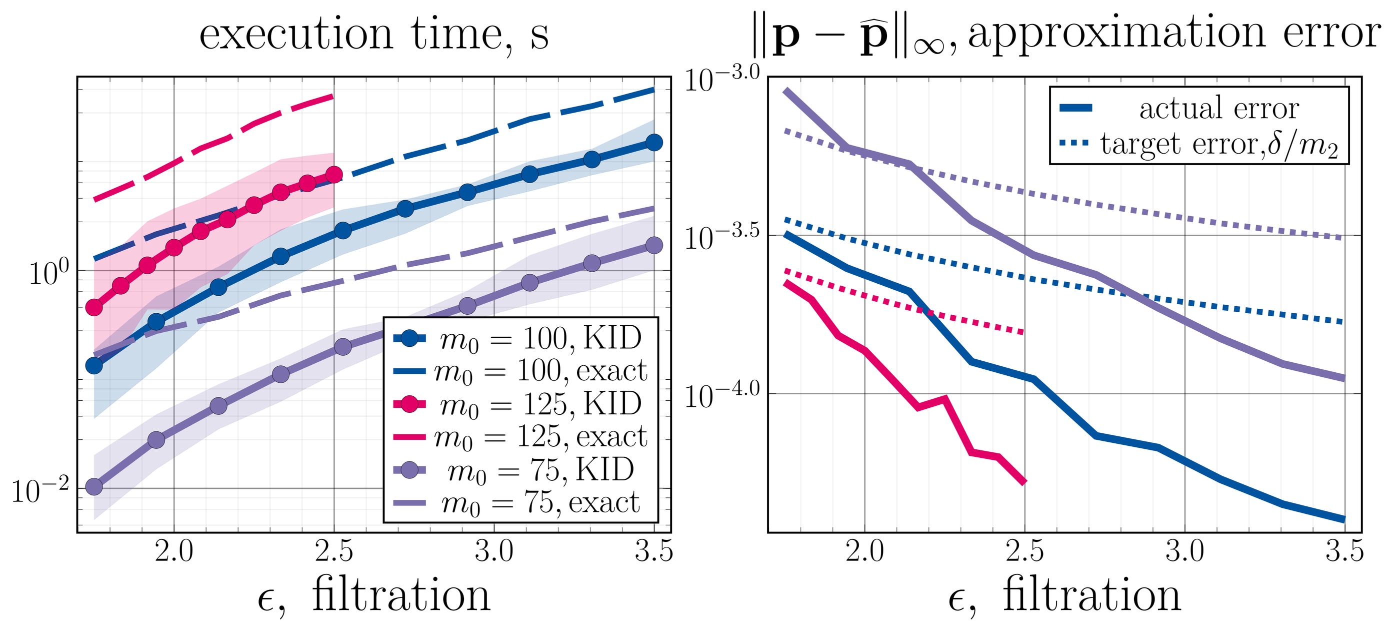
3. decompose using odd Chebyshev polynomials

$$\tilde{\mu}_j(\lambda \mid H) = \Sigma_{m=1}^{M//2} d_{mj} T_{2m+1}(\lambda)$$

$$\mathbb{E}\left[\mathcal{N}(0,1) \odot T_m(H) \times \mathcal{N}(0,1) \right] = \operatorname{diag} T_m(H)$$

4. use MonteCarlo-estimate for d_{mj}, $d_{m\bullet} = \operatorname{diag} T_m(H)$

Benchmark: Vietoris-Rips Filtration Complexes



Real-life datasets

dataset	GER time	KID time
email-enron	2.9s	0.31s
contact-hs	59.81s	0.0013s
NDC-classes	7.53s	0.246s
vegas-bars	NA	0.1179s
algebra	2.221s	0.433s

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