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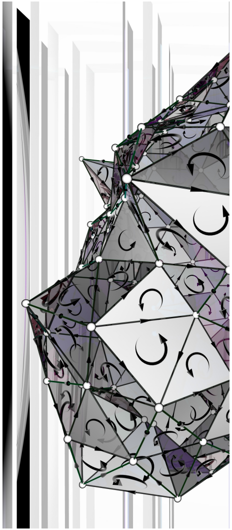

Topological Stability and Preconditioning of Higher-Order Laplacian Operators on Simplicial Complexes

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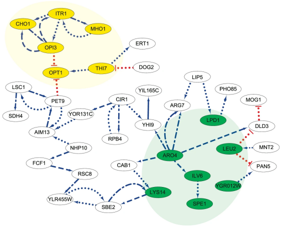
Overview

1. **Introduction**
   1.1 Relational Data and Graphs
   1.2 Higher-Order Models: Simplicial Complexes
   1.3 Homology Groups as Topological Descriptors
2. **Topological Stability of Simplicial Complexes**
   2.1 How to define topological stability?
   2.2 Principal Spectral Inheritance
   2.3 Gradient Flow for Spectral Matrix Nearness Problems
3. **Preconditioning for up-Laplacians**
   3.1 Joint solver for k-Laplacian
   3.2 Cholesky preconditioner
   3.3 Weakly Collapsible Complex
   3.4 Preconditioning by Subcomplex

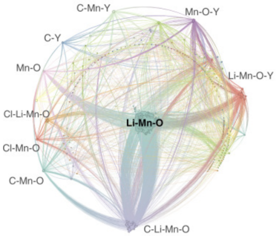
From Graphs to Simplicial Complexes: Algebra of Boundary Operators and Homology Groups

Networks of Interactions

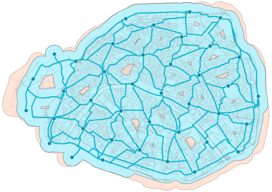
- **Relational Data** (*multi-agent systems with structured interactions*) are present in biology, neurology, chemistry, transportation and social networks, etc.
- Structure of the interactions can be induced by the **geometry** of the system or **functionality** of the interactions



(a) gene regulation



(b) chemical reactions



(c) bike paths in Paris

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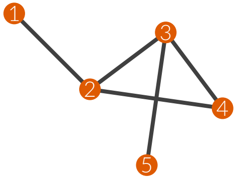
Graphs and matrices
Higher-order Models
Simplicial Complex
Homology Groups
Weighted Homology Groups
2 Stability of Homology Group
3 HeCS-preconditioning

Graphs are widely used models of multi-agent systems restricted to only **dyadic interactions**.

- graph $\mathcal{G} = (\mathcal{V}_0, \mathcal{V}_1)$ with $\mathcal{V}_0$ – set of **nodes** (agents) and $\mathcal{V}_1 \subseteq \mathcal{V}_0 \times \mathcal{V}_0$ – set of **edges**;
- associated matrices: *adjacency* (node vs node) A, *incidence* B (node → edge), *Laplacian* $L = BB^\top$, and *degree* matrix D, …

Used for

- dynamics on graphs: $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{y}$
- random walks: $\mathbf{p}_{t+1} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}} \mathbf{p}_t$
- label spreading: $\min \|\mathbf{y} - \mathbf{x}\| + \alpha \sum A_{ij} |x_i - x_j|^2$
- centrality measures: $\mathbf{c} = \frac{1}{\lambda} A^\top \mathbf{c}$



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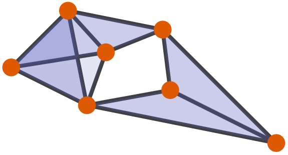
Higher-Order Models

Graph (classical)
pairwise interactions

Higher-order Models
non-linear relations

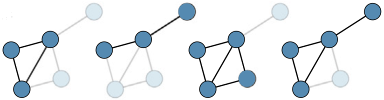


motifs , hypergraphs, simplicial complexes

Definition

Motifs are specific repeating subgraphs (e.g. triangles, 4-cycles, etc.).

- may promote label spreading or synchronization
- frequency signature for networks' classification
- require preexisting knowledge of the structures



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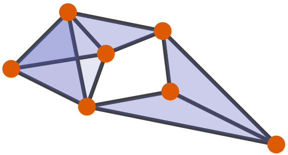
pairwise interactions

Higher-order Models
non-linear relations

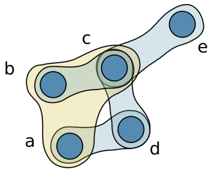


motifs , **hypergraphs**,
simplicial complexes

Definition

Hypergraph models allows interactions as any possible subset of nodes (**hyperedge**).

- much more general model
- lack of structure and restrictions complicate topological analysis
- may require tensor machinery with worse tractability



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Graph (classical)
pairwise interactions

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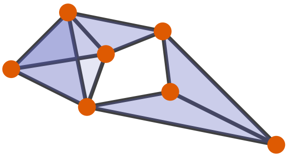
Higher-order Models
non-linear relations



motifs , hypergraphs,
simplicial complexes

Definition

Simplicial complex $\mathcal{K} = \{\sigma\}$ is a collection of nodal simplexes:

- each interaction in the system is a nodal simplex;
- each face of a simplex $\sigma \in \mathcal{K}$ also lies in $\mathcal{K}$.

$$\mathcal{K} = \underbrace{\mathcal{V}_0(\mathcal{K})}_{\text{nodes}}, \underbrace{\mathcal{V}_1(\mathcal{K})}_{\text{edges}}, \underbrace{\mathcal{V}_2(\mathcal{K})}_{\text{triangles}}, \ldots$$

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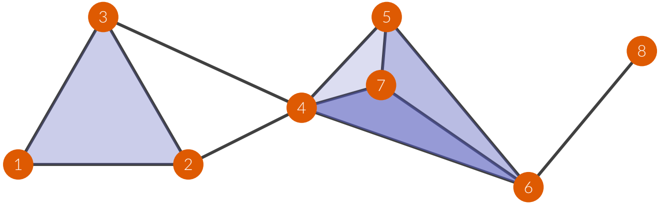

$$\mathcal{V}_0(\mathcal{K}) = \{[1],[2],[3],[4],[5],[6],[7],[8]\}, \qquad m_0 = 8$$

$$\mathcal{V}_1(\mathcal{K}) = \{[1,2],[1,3],[2,3],[2,4],[3,4],[4,5],$$
$$[4,6],[4,7],[5,6],[5,7],[6,7],[6,8]\}, \qquad m_1 = 12$$

$$\mathcal{V}_2(\mathcal{K}) = \{[1,2,3],[4,5,7],[4,6,7],[5,6,7]\}, \qquad m_2 = 4$$

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Definition

Topological properties of the simplicial complex $\mathcal{K}$ are described via **boundary operators** B_k:

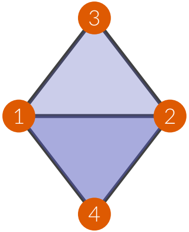
$$B_k : \text{ simplex } \sigma \longrightarrow \text{ boundary (faces) of } \sigma$$

Chain space C_k is a linear space of formal sums of simplexes from $\mathcal{V}_k(\mathcal{K})$:

- C_0 – space of states of nodes (e.g. labels);
- C_1 – space of edge flows;
- C_2 – space of states of triangles, etc.

Simplicial Complex and Topology
Boundary operators

$$B_k : C_k \mapsto C_{k-1}, \qquad B_k[v_1, \ldots v_{k+1}] = \sum_j (-1)^j [v_1, \ldots v_{j-1}, v_{j+1}, \ldots v_{k+1}]$$



B_1	12	13	14	23	24
1	-1	-1	-1	0	0
2	1	0	0	-1	-1
3	0	1	0	1	0
4	0	0	1	0	1

B_2	123	124
12	1	1
13	-1	0
14	0	-1
23	1	0
24	0	1

Fundamental Lemma of Topology: $B_k B_{k+1} = 0$.

Hodge Laplacians on Graphs
L.H. Lim

Definition (Homology groups)

Since $B_k B_{k+1} = 0$, the k-th homology group is defined as

$$\mathcal{H}_k = {}^{\ker B_k}\!\big/_{\operatorname{im} B_{k+1}} \cong \ker B_k \cap \ker B_{k+1}^\top = \ker \underbrace{\left(B_k^\top B_k + B_{k+1} B_{k+1}^\top\right)}_{L_k}$$

where L_k is the k-th graph Laplacian operator.

- $L_0 = B_1 B_1^\top$ – classical graph Laplacian operator
- $L_1 = B_1^\top B_1 + B_2 B_2^\top$ – 1- (Hodge) Laplacian operator
- $L_k^\uparrow = B_{k+1} B_{k+1}^\top$ – up-Laplacian, $L_k^\downarrow = B_k^\top B_k$ – down-Laplacian

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Hodge Decomposition $(k = 1)$:

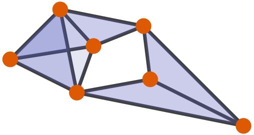
$$\mathbb{R}^{m_1} = \overbrace{\operatorname{im} B_1^\top \oplus \underbrace{\ker\left(B_1^\top B_1 + B_2 B_2^\top\right)}_{}}^{\ker B_2^\top} \!\!\underbrace{\oplus \operatorname{im} B_2}_{\ker B_1}$$

Each flow $\mathbf{x} \in C_1$ have three parts in the decomposition, $\mathbf{x} = \mathbf{y} + \mathbf{z} + \mathbf{h}$:

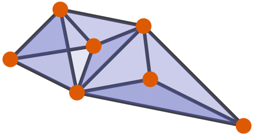
- $\mathbf{h} \in \ker L_1$ – **harmonic** part;
- note that the conjugate $B_1^\top [v_1, v_2] = [v_2] - [v_1]$, so $\mathbf{y} \in \operatorname{im} B_1^\top$ – **gradient** part
- similarly, $\mathbf{z} \in \operatorname{im} B_2$ – **curl** part

- elements of **ker** L_k correspond to the **k-dimensional holes** of $\mathcal{K}$;
- **dim ker** L_k = number of k-dimensional holes in $\mathcal{K}$.

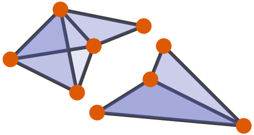
ker L_0 – **connected components**, **ker** L_1 – **1D holes**, **ker** L_2 – **3D voids**



dim ker $L_0 = 1$
dim ker $L_1 = 1$



dim ker $L_0 = 1$
dim ker $L_1 = 0$



dim ker $L_0 = 2$
dim ker $L_1 = 0$

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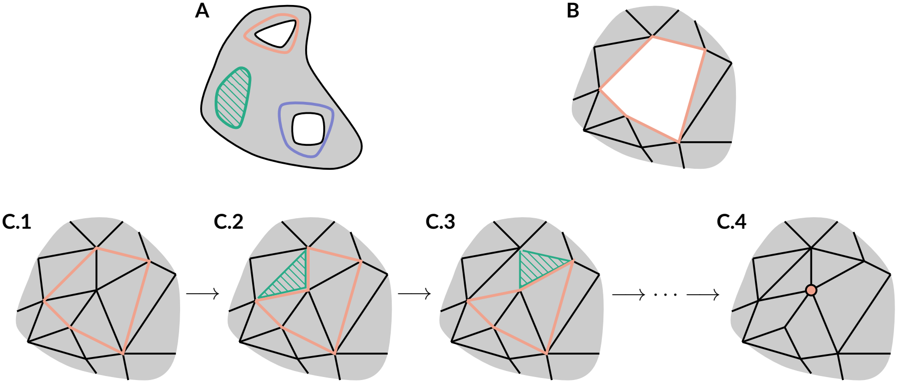

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- weight function for simplex of k-th order: $w_k : C_k \mapsto \mathbb{R}_+$;
- diagonal weight matrix $W_k = \mathbf{diag}\left(\left\{\sqrt{w_k(\sigma)}\right\}_{\sigma \in \mathcal{V}_k(\mathcal{K})}\right)$

$$B_k \quad \longrightarrow \quad \overline{B}_k = W_{k-1}^{-1} B_k W_k$$

Fundamental Lemma of Topology holds:

$$\overline{B}_k \overline{B}_{k+1} = \left(W_{k-1}^{-1} B_k W_k\right) \cdot \left(W_k^{-1} B_{k+1} W_{k+1}\right) = 0$$

Lemma (Weight impact on $\mathcal{H}_k$)

The dimension of the homology groups of $\mathcal{K}$ is not affected by the weights of its k-simplicies:

$$\dim \ker \overline{B}_k = \dim \ker B_k, \quad \dim \ker \overline{B}_k^\top = \dim \ker B_k^\top, \quad \dim \overline{\mathcal{H}}_k = \dim \mathcal{H}_k$$

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$$B_k \quad \longrightarrow \quad \overline{B}_k = W_{k-1}^{-1} B_k W_k$$

Fundamental Lemma of Topology holds:

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Common choices:

- **min-rule**: the weight of the triangle is the minimal weight of adjacent edges, $w_2(\tau) = \min\{ w_1(\sigma_1), w_1(\sigma_2), w_1(\sigma_3) \}$
- **product**: the weight of the triangle is the product of weights of adjacent edges, $w_2(\tau) = \sqrt[3]{w_1(\sigma_1) w_1(\sigma_2) w_1(\sigma_3)}$

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Topological Instability of Simplicial Complexes via Matrix Nearness Problems

Stability of the Homology Group

Problem Statement

Find the smallest (in weight) set of edges to eliminate in $\mathcal{K}$ such that:

$$\dim \overline{\mathcal{H}}_1(\widetilde{\mathcal{K}}) \geq \dim \overline{\mathcal{H}}_1(\mathcal{K}) + 1$$

create another hole in $\overline{\mathcal{H}}_1(\mathcal{K})$

$$\updownarrow$$

create another dimension in $\ker \overline{L}_1$

$$\updownarrow$$

push the smallest positive $\lambda_+ \in \sigma(\overline{L}_1)$ to 0

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Stability of the Homology Group

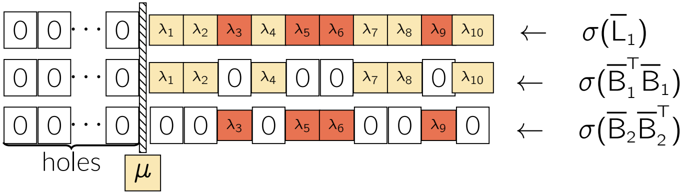
Principal Spectral Inheritance

Theorem (HOL's spectral inheritance)

Let $\overline{L}_k^{\text{down}} = \overline{B}_k^\top \overline{B}_k$ *and* $\overline{L}_k^{\text{up}} = \overline{B}_{k+1}\overline{B}_{k+1}^\top$ *(so* $\overline{L}_k = \overline{L}_k^{\text{down}} + \overline{L}_k^{\text{up}}$*). Then:*

1. $\sigma_+(\overline{L}_k^{\text{up}}) = \sigma_+(\overline{L}_{k+1}^{\text{down}})$, *where* $\sigma_+(\cdot)$ *denotes the set of positive eigenvalues;*
2. *for any* $\mu \in \sigma_+(\overline{L}_k)$, *either* $\mu \in \sigma_+(\overline{L}_k^{\text{up}})$ *or the corresponding eigenvector* $\vec{v} \in \ker \overline{L}_k^{\text{up}}$. *Similarly, for any* $\nu \in \sigma_+(\overline{L}_{k+1})$, *either* $\nu \in \sigma_+(\overline{L}_{k+1}^{\text{down}})$ *or the corresponding eigenvector* $\vec{u} \in \ker \overline{L}_{k+1}^{\text{down}}$, *and*

$$\overline{B}_k^\top \overline{B}_k \vec{v} = \mu \vec{v}, \qquad \overline{B}_{k+2}\overline{B}_{k+2}^\top \vec{u} = \nu \vec{u}\,.$$



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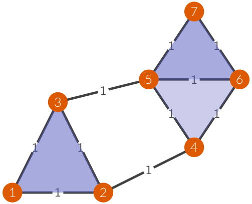
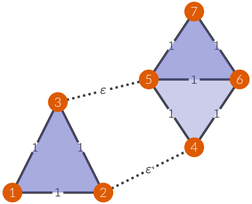
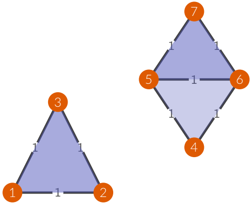


Connected $\mathcal{K}$
1 hole



Almost disconnected $\mathcal{K}$
1 hole



Disconnected $\mathcal{K}$
no holes

First positive $\lambda_+ \in \sigma(\overline{L}_1)$ may be inheritted from $\sigma(\overline{L}_0)$ and **does not relate** to the new hole in $\mathcal{H}_k$ (**homological pollution**).

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Creating another hole in $\overline{\mathcal{H}}_1 \iff$ push $\lambda_+ \in \sigma(\overline{L}_1^{up}) \to 0$, and avoid **homological pollution**.

combinatorial approach

find edges $E^- \subset \mathcal{V}_1(\mathcal{K})$ to eliminate such that

$$\dim \overline{\mathcal{H}}_1(\widetilde{\mathcal{K}}) \geq \dim \overline{\mathcal{H}}_1(\mathcal{K}) + 1$$

with

$$\sum_{e \in E^-} w_1(e) \to \min$$

$\longrightarrow$

continuous approach

perturb edges' weights

$$W_1 \to W_1 + \delta W_1$$

such that $\lambda_+(\delta W_1) = 0$ with

$$\|\delta W_1\| \to \min$$

spectral matrix nearness problem

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Problem: find *smallest* X such that $A + X$ has desired spectral properties.

Target eigenvalue and **functional**: for example, right-most eigenvalue $\lambda_{max}(A + X)$ or first non-zero eigenvalue $\lambda_+(A + X)$ with $F(X) = \frac{1}{2}\lambda^2_{max}(A + X)$.
Gradient flow: Integrate in the direction of anti-gradient of the target functional.

Lemma (Derivative of the eigenvalue)

$A(\tau)$ *has a unique eigenvalue* $\lambda(\tau)$ *that is analytic in a neighborhood of* τ_0, *with* $\lambda(\tau_0) = \lambda_0$

$$\dot{\lambda}(\tau_0) = \frac{1}{\mathbf{y}_0^* \mathbf{x}_0} \mathbf{y}_0^* \dot{A}(\tau_0) \mathbf{x}_0$$

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The edges' weight perturbation $W_1 \to W_1 + \delta W_1$ triggers a weight update for **nodes' weights** W_0 and **triangles weights** W_2:

- edge elimination ($w_1(e) + \delta w_1(e) = 0$) should trigger the elimination of all adjacent triangles, e.g. for $t = [e_1, e_2, e_3]$

$$w_2(t) \sim \min \{ w_1(e_1) + \delta w_1(e_1), w_1(e_2) + \delta w_1(e_2), w_1(e_3) + \delta w_1(e_3) \}$$

- vertex isolation should not trigger its elimination, e.g.

$$w_0(v) \sim \rho + \sum_{v \in e} \left(w_1(e) + \delta w_1(e) \right), \quad \rho > 0$$

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Let $\delta W_1 = \varepsilon E$, where ε is the **pertrubation size** and E, $\|E\| = 1$, is the **perturbation shape**.

Target functional:

$$F(\varepsilon, E) = \overbrace{\frac{1}{2}\lambda_+(\varepsilon, E)^2}^{\text{extend } \mathcal{H}_1} + \frac{\alpha}{2}\overbrace{\max\left(0, 1 - \frac{\mu_2(\varepsilon, E)}{\mu}\right)^2}^{\text{prevent pollution}}$$

where $\lambda_+(\varepsilon, E)$ is the smallest positive eigenvalue of perturbed $\overline{L}_1^{up}(\varepsilon, E)$ and $\mu_2(\varepsilon, E)$ is the algebraic connectivity of perturbed $\overline{L}_0(\varepsilon, E)$.

Gradient Flow for Steepest Descent

Let $E = E(t)$. Then:

$$\frac{d}{dt}F(\varepsilon, E(t)) = \varepsilon \left\langle \nabla_E F(e, E(t)), \dot{E} \right\rangle \Longrightarrow$$

steepest **monotone** descent

$$\dot{E} = -\nabla_E F(e, E(t))$$

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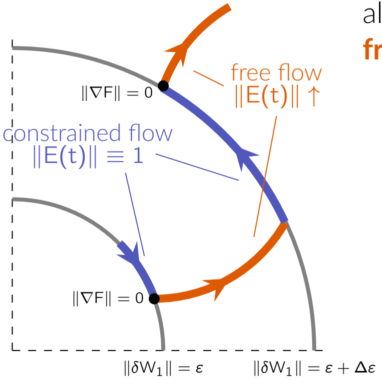

Given the intricate landscape of $F(\varepsilon, E)$, we alternate **constrained**, norm-preserving, and **free** gradient flows:

- **constrained**: $\dot{E} = -\nabla_E F(e, E(t)) + \kappa E(t)$ where κ is given by $\langle \dot{E}, E \rangle$;
- **free** flow: $\dot{E} = -\nabla_E F(e, E(t))$ untill $\|\varepsilon E(t)\| = \varepsilon + \Delta\varepsilon$;
- both flows use non-negativity projector $\mathbb{P}_+$ to avoid negative weights;
- the functional $F(\varepsilon, E)$ monotonically decreases.

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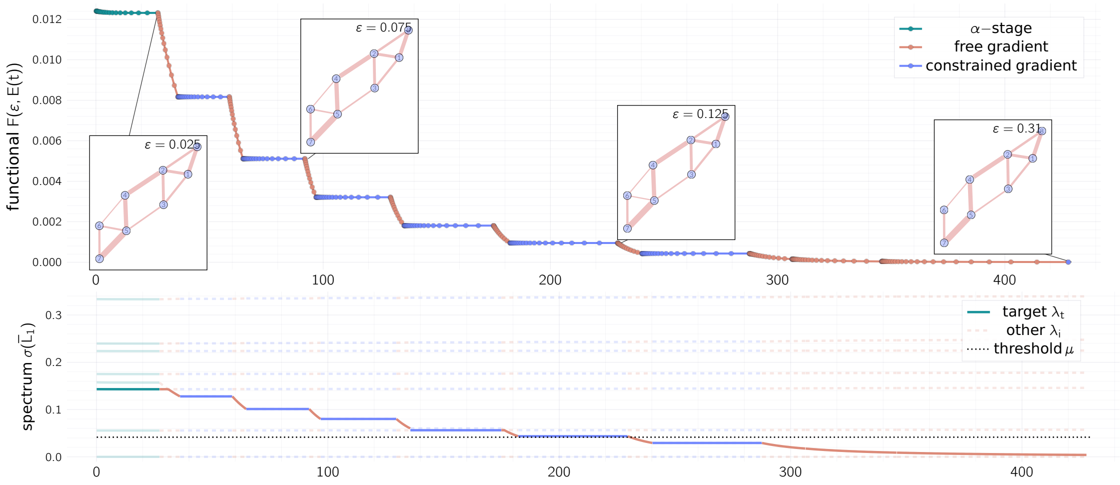
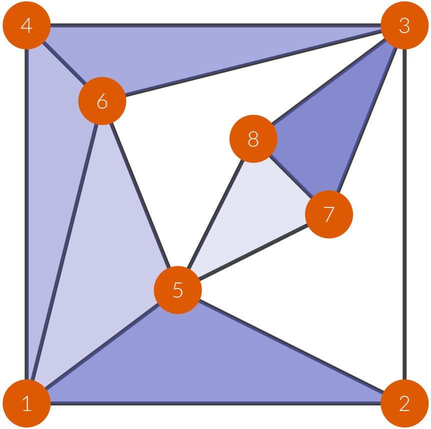
Illustrative example





- $(n-4)$ points are randomly thrown on the unit square;
- Delauney triangulation of sampled and corner points is calculated;
- edges randomly added or removed to reach the target sparsity ν;
- weights of the edges are randomly sampled, $w_i \sim U\left[\frac{1}{4}, \frac{3}{4}\right]$.

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Free and Constrained Stages
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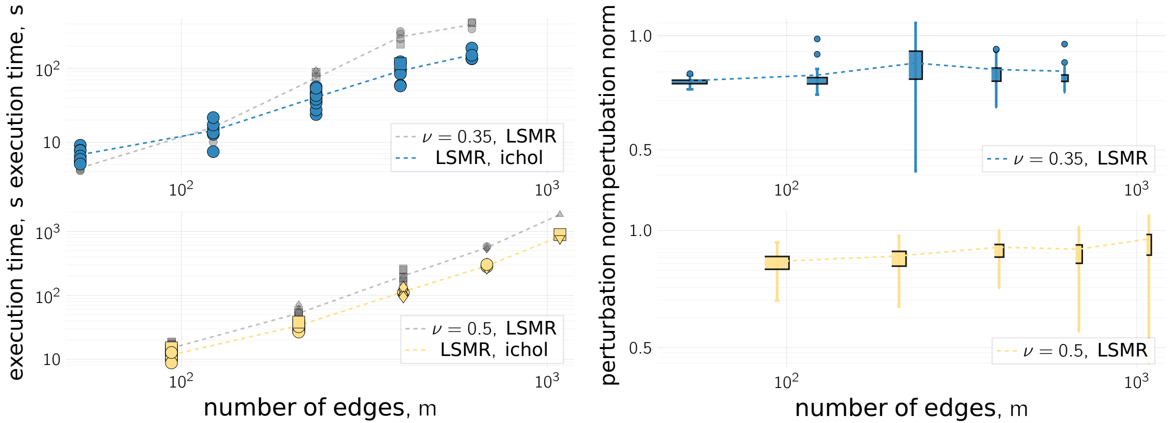
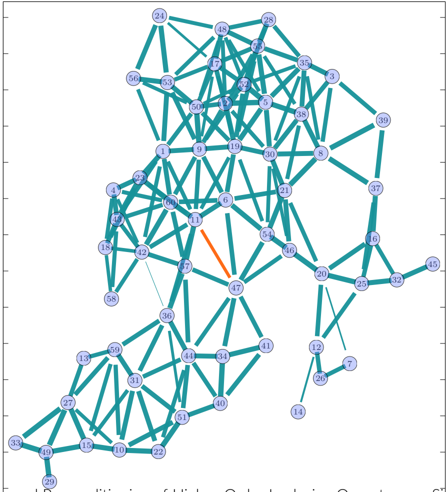
3 HeCS-preconditioning

Numerical benchmark
Triangulation

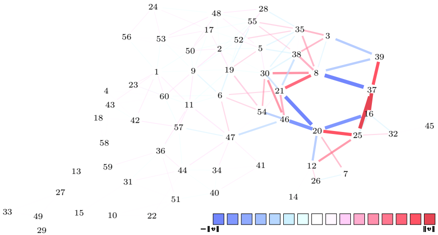


Real data example
Transportation Network

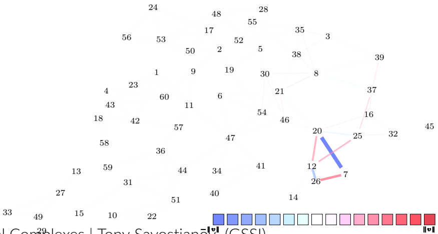
Bologna: Regional Network



1st original eigenflow



2nd original eigenflow



Cholesky-like Preconditioning via Heavy Collapsible Subcomplex for Laplacian Systems

Linear System with Hodge Laplacian

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Linear System for Hodge Laplacians

$$\begin{matrix} L_k \mathbf{x} = \mathbf{f} \\ \mathbf{x}, \mathbf{f} \perp \ker L_k \end{matrix} \iff \min_{\mathbf{x}} \| L_k \mathbf{x} - \mathbf{f} \|$$

- inside simplicial dynamics $\dot{\mathbf{x}} = L_k \mathbf{x} - \mathbf{f}$ (stationary point and implicit integrators)
- iterative solutions $\mathbf{x}_l = L_k \mathbf{x}_{l-1}$ for spectrum computations
- inside implicit graph neural networks, $\mathbf{x} = \phi \left(W \mathbf{x} L_k + B \right)$
- projections for **gradient** and **curl** components in Hodge decomposition

Theorem (Joint k-Laplacian solver)

The least-square problem $L_k \mathbf{x} = \mathbf{f}$ *can be reduced to a sequence of consecutive least-square problems for isolated up-Laplacians. Precisely,* $\mathbf{x}$ *is a solution of*

$$L_k \mathbf{x} = \mathbf{f} \qquad s.\ t. \qquad \mathbf{x}, \mathbf{f} \perp \ker L_k$$

if and only if it can be written as $\mathbf{x} = B_k^\top \mathbf{u} + \mathbf{x}_2$*, where:*

$$\widehat{\mathbf{u}} = \arg\min_{\mathbf{z}} \left\| L_{k-1}^{\uparrow} \mathbf{z} - B_k \mathbf{f}_1 \right\|, \quad \mathbf{u} = \arg\min_{\mathbf{z}} \left\| L_{k-1}^{\uparrow} \mathbf{z} - \widehat{\mathbf{u}} \right\|,$$

$$\mathbf{x}_2 = \arg\min_{\mathbf{y}} \left\| L_k^{\uparrow} \mathbf{y} - \mathbf{f}_2 \right\|$$

and $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$ *with* $\mathbf{f}_1 = B_k^\top \mathbf{z}_1$, $\mathbf{z}_1 = \arg\min_{\mathbf{z}} \left\| L_{k-1}^{\uparrow} \mathbf{z} - B_k \mathbf{f} \right\|$.

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Consequence

Solution of the "whole" Laplacian system $L_k \mathbf{x} = \mathbf{f}$ can be reduced to solving only up-Laplacian systems $L_k^{\uparrow} \mathbf{x} = \mathbf{f}$.

- both L_k and $L_k^{\uparrow}$ are sparse, so iterative solvers (CGLS, LSMR, etc.) can benefit from fast `matvec` operation;
- convergence of such methods are primarily determined by the **condition number** $\kappa_+(L_k^{\uparrow})$, or, specifically:

$$\|\mathbf{x}_N - \mathbf{x}^*\|_{L_k^{\uparrow}} \leq 2 \left(\frac{\sqrt{\kappa_+(L_k^{\uparrow})} - 1}{\sqrt{\kappa_+(L_k^{\uparrow})} + 1} \right)^N \|\mathbf{e}_0\|_{L_k^{\uparrow}}$$

- since all Laplacians are naturally singular, we use $\kappa_+(L_k^{\uparrow}) = \frac{\sigma^+_{\max}(L_k^{\uparrow})}{\sigma^+_{\min}(L_k^{\uparrow})}$

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Definition (Sparse Simplicial Complex)

We assume $\mathcal{K}$ to be a **sparse simplicial complex** (in comparison with always sparse Laplacian operators) at the $k-$th order, if

$$m_k = \mathcal{O}(m_{k-1} \log m_{k-1})$$

analogously to the standard graph case.

Problem

In order to reduce $\kappa_+(L_k^{\uparrow})$, we want to move:

$$\min_{\mathbf{x}} \|L_k^{\uparrow}\mathbf{x} - \mathbf{f}\| \longrightarrow \min_{\mathbf{x}} \left\| \left(C^{\dagger} L_k^{\uparrow} C^{\top\dagger}\right)\left(C^{\top}\mathbf{x}\right) - C^{\dagger}\mathbf{f} \right\|$$

such that the transition is bijective and $C^{\dagger}$ is cheap.

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If C is lower-triangular, the pseudo-inverse is cheap and C is known as **Cholesky preconditioner**. It is computed along **Schur complements**:

$$S_i = S_{i-1} - \frac{1}{\alpha_i} S_{i-1}\delta_i\delta_i^\top S_{i-1}^\top$$

$$\mathbf{c}_i = \frac{1}{\sqrt{\alpha_i}} S_{i-1}\delta_i$$

$$\alpha_i = \delta_i^\top S_{i-1}\delta_i$$

With these definitions, the Cholesky factor C such that $A = CC^\top$ is formed by the columns $\mathbf{c}_i$, namely

$$C = [\mathbf{c}_1\, \mathbf{c}_2\, \ldots\, \mathbf{c}_n]$$

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- computation of exact Schur complements S_i is expensive
- one aims to leverage underlying simplicial complex $\mathcal{K}$ to speed it up

Lemma (Rank-1 decomp.): $L_k^{\uparrow} = \sum_{t \in \mathcal{V}_k(\mathcal{K})} L_k^{\uparrow}(t) = \sum_{t \in \mathcal{V}_k(\mathcal{K})} w(t) \mathbf{e}_t \mathbf{e}_t^{\top}$

- $w(t) = [W_k^2]_{tt}$ is the weight of the simplex t
- $\mathbf{e}_t$ is the t-th column of the matrix B_k

Then, for $L_1^{\uparrow}$:

$$S_1 = \underbrace{\sum_{t | 1 \notin t} w(t) \mathbf{e}_t \mathbf{e}_t^{\top}}_{\text{remainder of } L_1^{\uparrow} \text{ without edge } 1} + \underbrace{\frac{1}{2\Omega_{\{1\}|\varnothing}} \sum_{\substack{t_1 | 1 \in t_1 \\ t_2 | 1 \in t_2}} w(t_1) w(t_2) [\mathbf{e}_{t_1} - \mathbf{e}_{t_2}][\mathbf{e}_{t_1} - \mathbf{e}_{t_2}]^{\top}}_{\text{clique term, not a Laplacian}}$$

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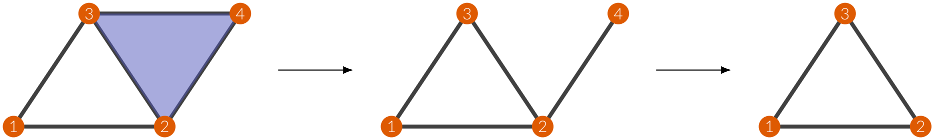
The simplex $\sigma \in \mathcal{K}$ is free if it is a face of exactly one simplex $\tau = \tau(\sigma) \in \mathcal{K}$ of higher order (maximal face).

The collapse $\mathcal{K}\backslash\{\sigma\}$ of $\mathcal{K}$ at a free simplex σ is the operation of reducing $\mathcal{K}$ to $\mathcal{K}'$, where $\mathcal{K}' = \mathcal{K} - \sigma - \tau$.

Simplex $\mathcal{K}$ is called **collapsible** if it can be reduced to a single node.



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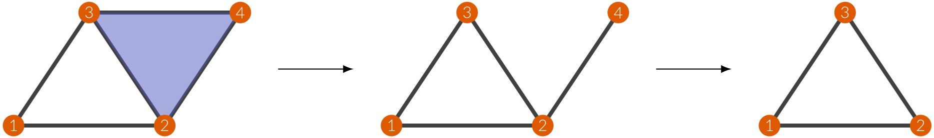
Weak Collapsibility

General collapsibility is *NP-hard* and too demanding for our purposes (to eliminate cyclic terms).

Definition (Weakly collapsible complex)

A simplicial complex $\mathcal{K}$ restricted to its 2-skeleton is called **weakly collapsible**, if there exists a collapsing sequence Σ_1 such that the simplicial complex $\mathcal{L} = \mathcal{K} \backslash \Sigma_1$ has no simplices of order 2, i.e. $\mathcal{V}_2(\mathcal{L}) = \varnothing$ and $L_1^{\uparrow}(\mathcal{L}) = 0$.



Weak collapsibility can be consistently checked by `GREEDY ALGORITHM` and polynomially solvable($\mathcal{O}(m_1)$).

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Lemma (Exact Solver for Collapsible Complexes)

Let the simplicial complex $\mathcal{K}$ *be weakly collapsible through the collapsing sequence* Σ *and the corresponding sequence of maximal faces* $\mathbb{T}$*. Let the permutation matrices of the two sequences be* P_Σ *and* $P_\mathbb{T}$*, i.e. such that* $[P_\Sigma]_{ij} = 1 \iff j = \sigma_i$*, and similarly for* $P_\mathbb{T}$*. Then* $C = P_\Sigma \overline{B}_2 P_\mathbb{T}$ *is an exact Cholesky multiplier for* $P_\Sigma L_1^\uparrow(\mathcal{K}) P_\Sigma^\top$*, i.e.* $P_\Sigma L_1^\uparrow(\mathcal{K}) P_\Sigma^\top = CC^\top$*.*

Idea

Find a weakly collapsible **subcomplex** $\mathcal{L} \subseteq \mathcal{K}$ and use its Cholesky multiplier C as preconditioner.

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Preconditioning by Subcomplex
How to find the best subcomplex?

Definition (Subsampling Matrix)

Diagonal matrix Π is called subsampling matrix if $\Pi_{ii} = 1$ only if the i-th triangle is included in subcomplex $\mathcal{L}$.

Lemma (Optimal Weight Choice)

Triangles should be sampled in $\mathcal{L}$ with the same weight as the original $\mathcal{K}$.

Theorem (Preconditioning by Subcomplex)

Let $\mathcal{L}$ be a weakly collapsible subcomplex of $\mathcal{K}$ defined by the subsampling matrix Π and let C be a Cholesky multiplier of $L_1^{\uparrow}(\mathcal{L})$. Then the conditioning of the symmetrically preconditioned $L_1^{\uparrow}$ is given by:

$$\kappa_+\left(C^{\dagger} P_{\Sigma} L_1^{\uparrow} P_{\Sigma}^{\top} C^{\dagger\top}\right) = \left(\kappa_+\left(\left(S_1 V_1^{\top} \Pi\right)^{\dagger} S_1\right)\right)^2 = \left(\kappa_+(\Pi V_1)\right)^2,$$

where V_1 forms the orthonormal basis on $\operatorname{im} \overline{B}_2^{\top}$.

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Preconditioning quality is determined by $\kappa_+(\Pi V_1)$:

- Note that $\operatorname{im} V_1 = W_2 \operatorname{im} B_2^\top$
- Rows in V_1 are scaled by the weights of the triangles
- Multiplication by Π cancels rows in V_1 for the eliminated triangles
- Good choice of Π: *eliminate triangles with the smallest edges*

Subcomplex $\mathcal{L}$ should:

1. have the same set of nodes and edges;
2. subsample triangles, $\mathcal{V}_2(\mathcal{L}) \subseteq \mathcal{V}_2(\mathcal{K})$;
3. be weakly collapsible through some collapsing sequence Σ and sequence of maximal faces $\mathbb{T}$;
4. have the same 1-homology as $\mathcal{K}$, that is $\ker L_1(\mathcal{K}) = \ker L_1(\mathcal{L})$ (bijectivity);
5. have the highest possible total weight to improve preconditioning.

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Preconditioning by Subcomplex
Heavy Collapsible Subcomplex

Algorithm 5 HEAVY_SUBCOMPLEX$(\mathcal{K}, W_2)$: construction a heavy collapsible subcomplex

Require: the original complex $\mathcal{K}$, weight matrix W_2

```
L ← ∅, T ← ∅                                        ▷ initial empty subcomplex
while there is unprocessed triangle in V_2(K) do
    t ← nextHeaviestTriangle(K, W_2)                ▷ e.g. iterate through V_2(K) sorted by weight
    if L ∪ {t} is weakly collapsible then           ▷ run GREEDY_COLLAPSE(L ∪ {t}) (Algorithm 4)
        L ← L ∪ {t}, T ← [T t]                      ▷ extend L by t
    end if
end while
return L, T, Σ                                      ▷ here Σ is the collapsing sequence of L
```

- we assume $\mathcal{K}$ to be sparse, $m_2 = \mathcal{O}(m_1 \log m_1)$;
- $\mathcal{K}$ has a disbalanced weight profile for triangles (e.g. generated `minrule`), so a dominating heavy subcomplex is more probable;
- algorithmic complexity of `HeCS` preconditioning is $\mathcal{O}(m_1 m_2)$.

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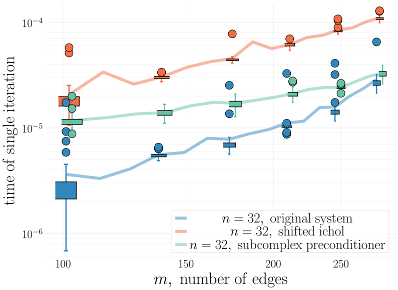
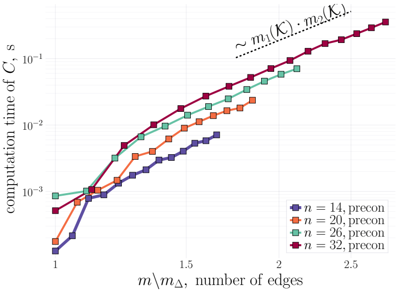


(a) Single iteration timing: the average time of **`matvec`** computation for the original system (blue), shifted **`ichol`** (orange) and **`HeCS`** preconditioner (green).



(b) Computation time for the heavy subcomplex preconditioner in case of enriched triangulations on m_0 vertices

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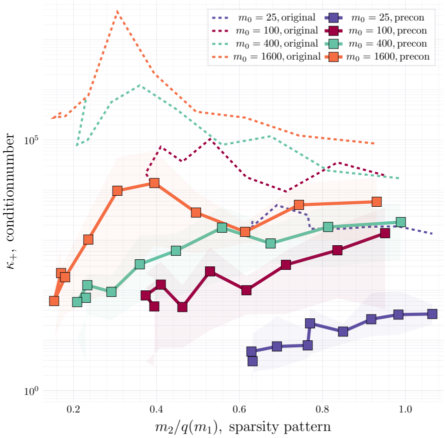

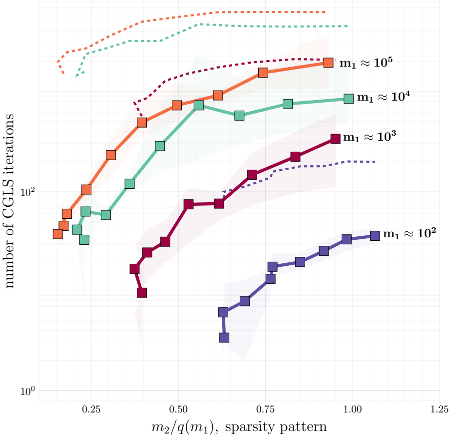

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Thank you for attention!

Topological Stability

paper on arXiV

code on Github

HeCS - Preconditioning

paper on arXiV

code on Github

Personal Page: further materials on https://antsav.me/
Comp@GSSI: our group at https://num-gssi.github.io/

Break the screen in case of . . .

Well, just break it,
we will figure it out later!

- main computational load: first non-zero eigenvalue of both $\overline{L}^{\uparrow}_1(\varepsilon, E)$ and $\overline{L}_0(\varepsilon, E)$
- both operators are of form $A^\top A$ (with $A = \overline{B}_2$ or $A = \overline{B}_1^\top$);
- corresponding optimization problem: $\min_{\mathbf{x} \perp \ker A} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$
- requires sparse singular value solver based on a Krylov subspace scheme for the pseudo inverse of $A^\top A$

L-Sq Problems (solve with preconditioned LSMR):

$$\min_{\mathbf{x}} \|\overline{L}_1^{up}(\varepsilon, E)\mathbf{x} - \mathbf{b}\|, \qquad \min_{\mathbf{x}} \|\overline{L}_0(\varepsilon, E)\mathbf{x} - \mathbf{b}\| ,$$

Idea: fix the preconditioner along the flow.

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Theorem (Simplicial Sparsification)

For any $\varepsilon > 0$, *a sparse simplicial complex* $\mathcal{L}$ *can be sampled from* $\mathcal{K}$ *as follows:*

1. *compute the probability measure* $\mathbf{p}$ *on* $\mathcal{V}_{k+1}(\mathcal{K})$ *proportional to the generalized resistance vector* $\mathbf{r} = \operatorname{diag}\left(B_{k+1}^\top (L_k^\uparrow)^\dagger B_{k+1}\right)$;
2. *sample* q *simplices* τ_i *from* $\mathcal{V}_{k+1}(\mathcal{K})$ *according to the probability measure* $\mathbf{p}$, *where* q *is chosen so that* $q(m_k) \geq 9C^2 m_k \log(m_k/\varepsilon)$, *for some absolute constant* $C > 0$;
3. *form a sparse simplicial complex* $\mathcal{L}$ *with all the sampled simplexes of order* k *and all its faces with the weight* $\frac{w_{k+1}(\tau_i)}{q(m_k)\mathbf{p}(\tau_i)}$; *weights of repeated simplices are accumulated.*

Then, with probability at least $1/2$, *the up-Laplacian of the sparsifier* $\mathcal{L}$ *is* ε*-close to the original one, i.e. it holds* $L_k^\uparrow(\mathcal{L}) \underset{\varepsilon}{\approx} L_k^\uparrow(\mathcal{K})$.

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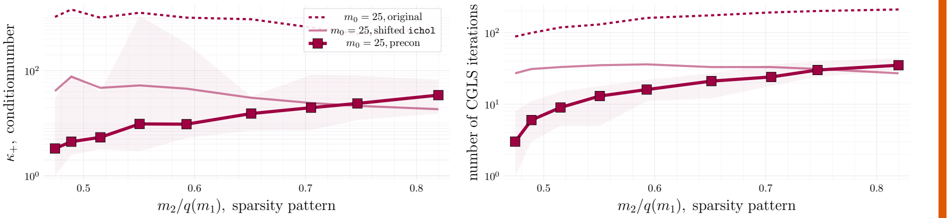
HeCS
Shifted `ichol` comparison

Assuming U is an orthogonal basis of $\ker L_1^{\uparrow}$, one can move to $L_1^{\uparrow} \to L_1^{\uparrow} + \alpha U U^{\top}$, which can be preconditioned by non-singular methods. Specifically, we use $C_\alpha = \texttt{ichol}(L_1^{\uparrow} + \alpha U U^{\top})$.



U can be formed directly using the vectors $B_1^{\top}\mathbf{x}$, $\mathbf{x} \in (\mathbf{1})^{\perp}$, when $\mathcal{K}$ has trivial 0- and 1-homology.

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